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Chebyshev type lattice path weight polynomials by a constant term method

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Abstract

We prove a constant term theorem which is useful for finding weight polynomials for Ballot/Motzkin paths in a strip with a fixed number of arbitrary ‘decorated’ weights as well as an arbitrary ‘background’ weight. Our CT theorem, like Viennot’s lattice path theorem from which it is derived primarily by a change of variable lemma, is expressed in terms of orthogonal polynomials which in our applications of interest often turn out to be non-classical. Hence, we also present an efficient method for finding explicit closed-form polynomial expressions for these non-classical orthogonal polynomials. Our method for finding the closed-form polynomial expressions relies on simple combinatorial manipulations of Viennot’s diagrammatic representation for orthogonal polynomials. In the course of the paper we also provide a new proof of Viennot’s original orthogonal polynomial lattice path theorem. The new proof is of interest because it uses diagonalization of the transfer matrix, but gets around difficulties that have arisen in past attempts to use this approach. In particular we show how to sum over a set of implicitly defined zeros of a given orthogonal polynomial, either by using properties of residues or by using partial fractions. We conclude by applying the method to two lattice path problems important in the study of polymer physics as the models of steric stabilization and sensitized flocculation.

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1. Introduction and definitions

As is well known to mathematical physicists the form of the solution to a problem is often more important than its existence. Such is the case in this paper. Determining the generating function of Motzkin path weight polynomials in a strip was solved by a theorem due to Viennot [29] (see theorem 1 below—hereafter referred to as Viennot’s theorem). In the applications

discussed below what is required are the weight polynomials themselves. Whilst these can be written as a Cauchy integral of the generating function, this form of the solution is of little direct use for our applications. To this end we have derived a related form of the generating function given by theorem 2 (hereafter referred to as the constant term, or CT, theorem). The CT theorem is certainly well suited to extracting the weight polynomials for Chebyshev type problems (section 7) and as a starting point for their asymptotic analysis. This route to the asymptotic analysis starts with the constant term expression and replaces it with a contour integral which can be tackled by the method of steepest descents, the details of the particular form of the integrand being important to the utility of this method—see [25] for such an application. It is very useful that the ‘order t ’ factor appears in the numerator (see (13)), rather than in the denominator as occurs with the Cauchy form (see (15)).

The CT theorem is proved, as we shall show below, by starting with Viennot’s theorem and using a ‘change of variable’ lemma (see lemma 5.1). We will also provide a new proof of Viennot’s theorem that is based on diagonalizing the associated Motzkin path transfer matrix. The latter proof is included as it rather naturally leads to the CT theorem. It also has some additional interest as it has several combinatorial connections [7]. For example, a combinatorial interpretation of what has previously appeared only as a change of variable to eliminate a square root in Chebyshev polynomials turns out to be the generating function of binomial paths. Another connection is a combinatorial interpretation of the Bethe Ansatz [1] as determining the signed set of path involutions, as for example, in the involution of Gessel and Viennot [20] in the many path extension [8].

Two classes of applications for which the CT theorem is certainly suited are the asymmetric simple exclusion process (ASEP) and directed models of polymers interacting with surfaces. For the ASEP the problem of computing the stationary state, and hence in finding phase diagrams for associated simple traffic models, can be cast as a lattice path problem [2–4, 6, 10, 14, 16, 18]. For the ASEP model the path problem required is actually a half-plane model with two weights associated with the lower wall (the upper wall is sent to infinity to obtain the half plane.) [10]. In chemistry the lattice paths are used to model polymers in solution [15]—for instance in the analysis of steric stabilization and sensitized flocculation [11, 12].

In the application section of this paper we find explicit expressions for the partition function—or weight polynomial—of the DiMazio and Rubin polymer model [17]. This model was first posed in 1971 and has Boltzmann weights associated with the upper and lower walls of a strip containing a path. The wall weights model the interaction of the polymer with the surface. The solution given in this paper is an improvement on that published in [7]. Previous results on the DiMazio and Rubin model have only dealt with special cases of weight values, for example, the case where a relationship exists between the Boltzmann weights [9]. We also present a natural generalization of the DiMazio and Rubin weighting which may have application to models of polymers interacting with colloids [24], where the interaction strength depends on the proximity to the colloid.

In order to use the CT theorem both the ASEP and polymer models require explicit expressions for ‘perturbed’ Chebyshev orthogonal polynomials. Their computation is addressed by our third theorem, theorem 3.

1.1. Definitions and Viennot’s theorem

Consider length t lattice paths, $p = v_0 v_1 v_2 \dots v_t$, in a height L strip with vertices $v_i \in \mathcal{S} = \mathbb{Z}_{\geq 0} \times \{0, 1, \dots, L\}$, such that the edges $e_i := v_{i-1} v_i$ satisfy $v_i - v_{i-1} \in \{(1, 1), (1, 0), (1, -1)\}$. An edge e_i is called *up* if $v_i - v_{i-1} = (1, 1)$, *across* if $v_i - v_{i-1} = (1, 0)$ and *down* if $v_i - v_{i-1} = (1, -1)$. A vertex $v_i = (x, y)$ has *height* y . Weight the edges according

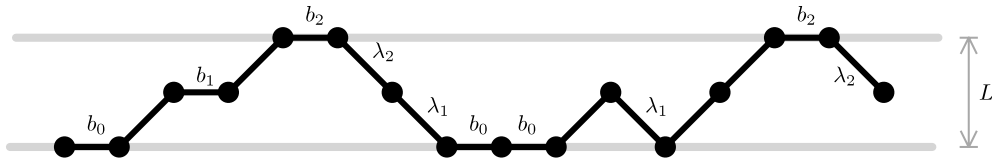


Figure 1. An example of a lattice path of length 15, in a strip of height $L = 2$, with weight $b_0^3 b_1 b_2^2 \lambda_1^2 \lambda_2^2$ and starting at $y' = 0$ and ending at $y = 1$.

to the height of their left vertex using

$$w(e_i) = \begin{cases} 1 & \text{if } e_i \text{ is an up edge} \\ b_k & \text{if } e_i \text{ is an across edge with } v_{i-1} = (i-1, k) \\ \lambda_k & \text{if } e_i \text{ is a down edge with } v_{i-1} = (i-1, k). \end{cases} \quad (1)$$

The weight of the path, $w(p)$, is defined to be the product of the weights of the edges, i.e. for path $p = v_0 v_1 \dots v_t$,

$$w(p) = \prod_{i=1}^t w(e_i). \quad (2)$$

Such weighted paths are then enumerated according to their length with *weight polynomial* defined by

$$Z_t(y', y; L) := \sum_p w(p), \quad (3)$$

where the sum is over all paths of length t , confined within the strip of height L , y' is the height of the initial vertex of the path and y is the height of the final vertex of the path. An example of such a path is shown in figure 1. These paths are weighted and confined elaboration's of Dyck paths, Ballot paths and Motzkin paths—for enumeration results on these classic paths see, for example, [22]. Once non-constant weights are added, many classical techniques do not (obviously) apply.

This work focuses upon solving the enumeration problem for the types of weighting in which a small number of weights take on distinguished values called ‘decorations’, and the rest of the edges have constant ‘background weights’ of just one of two kinds, accordingly as the step is an across step or a down step.

We introduce notation to describe the positions of the decorated edges: let $\mathcal{D}_b \subseteq \{0, \dots, L\}$ and $\mathcal{D}_\lambda \subseteq \{1, \dots, L\}$ be sets of integers called respectively *decorated across-step heights* and *decorated down-step heights*. Then paths are weighted as in equation (1), with

$$b_i = \begin{cases} b & \text{if } i \notin \mathcal{D}_b \\ b + \hat{b}_i & \text{if } i \in \mathcal{D}_b \end{cases} \quad (4a)$$

$$\lambda_i = \begin{cases} \lambda & \text{if } i \notin \mathcal{D}_\lambda \\ \lambda + \hat{\lambda}_i & \text{if } i \in \mathcal{D}_\lambda \end{cases} \quad (4b)$$

where b and λ will be called *background weights* and \hat{b}_i and $\hat{\lambda}_i$ will be called *decorations*.

The generating function for the weight polynomials is given in terms of orthogonal polynomials by a theorem due to Viennot [29, 30]—see also [19, 21]

Theorem 1 ([29]). *The generating function of the weight polynomial (3) is given by*

$$M_L(y', y; x) := \sum_{t \geq 0} Z_t(y', y; L) x^t = x^{Y-Y'} \frac{R_{Y'}(x) h_{y',y} R_{L-Y}^{(y+1)}(x)}{R_{L+1}(x)}, \tag{5}$$

where $h_{y',y} = \prod_{y < l \leq y'} \lambda_l$ if $y' > y$ and $h_{y',y} = 1$ if $y' \leq y$, $Y' = \min\{y', y\}$ and $Y = \max\{y', y\}$. The polynomials $R_k(x)$ are the reciprocal polynomials and $R_k^{(j)}(x)$ are the shifted reciprocal polynomials defined by

$$R_k(x) := x^k P_k(1/x) \quad \text{and} \quad R_k^{(j)}(x) := R_k(x) \Big|_{\substack{b_i \rightarrow b_{i+j} \\ \lambda_i \rightarrow \lambda_{i+j}}}, \tag{6}$$

where the orthogonal polynomials $P_k(x)$ satisfy the standard three-term recurrence [13, 28]

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x - b_0. \tag{7}$$

This theorem may be proved in several ways: as the ratio of determinants—see [27] section 4.7.2, by continued fractions [19, 21] or by heaps of monomers and dimers [31]. In section 4, we provide a new proof that uses diagonalization of the transfer matrix of the Motzkin paths.

2. A constant term theorem

Our main result is stated as a constant term of a particular Laurent expansion. Since the constant term method studied in this paper depends strongly on the choice of Laurent expansion, we briefly recall a few simple facts about Laurent expansions. Since we only consider rational functions we restrict our discussion to them. A Laurent expansion of a rational function about a point $z = z_i$ is of the form $\sum_{n \geq n_0} a_n (z - z_i)^n$. The coefficients a_n depends on the chosen point z_i and the annulus of convergence. Furthermore, the nature of n_0 generally depends on three factors: (i) $n_0 \geq 0$ (i.e. the series is a Taylor series) if the annulus contains no singular points, (ii) $n_0 < 0$ is finite if the inner circle of the annulus contains only a non-essential singularity at z_i and (iii) $n_0 = -\infty$ if the inner circle contains at least one other singularity at $z \neq z_i$ or an essential singularity.

In this paper, we only need case (ii), with $z_i = 0$ and the series convergent in the annulus closest to the origin. Thus, the constant term is defined as follows.

Definition 1. *Let $f(z)$ be a complex-valued function with Laurent expansion of the form*

$$f(z) = \sum_{n=n_0}^{\infty} a_n z^n \tag{8}$$

with $n_0 \in \mathbb{Z}$. Then the constant term in z of $f(z)$ is

$$CT_z[f(z)] = \begin{cases} a_0 & \text{if } n_0 \leq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

This is, of course, just the residue of $f(z)/z$ at $z = 0$. Note, the form of the Laurent expansion given in (8) uniquely specifies that it corresponds to that Laurent expansion of $f(z)$ that converges in the innermost annulus that is centred at the origin.

Our main result gives the weight polynomial for Motzkin paths in a strip as the constant term of a rational function constructed from Laurent polynomials. The Laurent polynomials

we use, $L_k^{(j)}(\rho)$, are defined in terms of the conventional (shifted) orthogonal polynomials, $P_k^{(j)}(x)$, by the simple substitution

$$L_k^{(j)}(\rho) := P_k^{(j)}(x(\rho)) \tag{10}$$

with

$$x(\rho) = \rho + b + \lambda\rho^{-1}. \tag{11}$$

The orthogonal polynomials $P_k(x) = P_k^{(0)}(x)$ satisfy the standard three-term recurrence (7) which, for the shifted polynomials $P_k^{(j)}(x)$, becomes

$$\begin{aligned} P_k^{(j)}(x) &= (x - b_{k+j-1})P_{k-1}^{(j)}(x) - \lambda_{k+j-1}P_{k-2}^{(j)}(x), & k \geq 2 \\ P_1^{(j)}(x) &= x - b_j \\ P_0^{(j)}(x) &= 1 \end{aligned} \tag{12}$$

and $\lambda_k \neq 0 \forall k$. We now state our principal theorem.

Theorem 2 (Constant term). *Let $Z_t(y', y; L)$ be the weight polynomial for the set of Motzkin paths with initial height y' , final height y , confined in a strip of height L and weighted as specified in equations (4). Then*

$$Z_t(y', y; L) = \text{CT}_\rho \left[\left(\rho + b + \frac{\lambda}{\rho} \right)^t \frac{L_{Y'}(\rho) h_{y',y} L_{L-Y}^{(Y+1)}(\rho)}{L_{L+1}(\rho)} \left(\frac{\lambda}{\rho} - \rho \right) \right], \tag{13}$$

with $Y' = \min\{y', y\}$, $Y = \max\{y', y\}$,

$$h_{y',y} = \begin{cases} \prod_{y < l \leq y'} \lambda_l & \text{if } y' > y \\ 1 & \text{otherwise.} \end{cases} \tag{14}$$

and the Laurent polynomials $L_k^{(j)}$ given by (10).

The form of this constant term expression should be carefully compared with that arising from Viennot’s theorem when used in conjunction with the standard Cauchy constant term form for the weight polynomial (for $y' \leq y$),

$$Z_t(y', y; L) = \text{CT}_x \left[\frac{1}{x^{t+1}} x^{y-y'} \frac{R_{y'}(x) R_{L-y}^{(y+1)}(x)}{R_{L+1}(x)} \right]. \tag{15}$$

In particular, whilst (13) is obtained by a ‘change of variable’, it is not obtained by simply replacing every occurrence of x by $1/(\rho + b + \lambda\rho^{-1})$ in (15) as was done to define the Laurent polynomials $L_k^{(j)}$. We provide some explanation of this as it is not commonly known that when changing variables in a constant term expression an additional factor is introduced. This factor arises from a derivative of the change of variable expression—this is explained in detail in the proof of this theorem—see section 5. In this case it is the factor $(\frac{\lambda}{\rho} - \rho)$. If one thinks of the constant term as a residue (see (41)) then clearly it can be expressed as a contour integral and hence a change of variable in the contour integral obviously has an additional derivative factor. However, it is not as simple as that, as one still has to address what happens to the location of the contour under the variable change. In fact, as detailed in the proof, it is only if the change of variable expression satisfies certain conditions (see lemma 5.1) can the effect of the variable change on the contour be ignored.

It is a useful exercise to compare the difference in effort in computing a *general expression* for $Z_t(y', y; L)$ in the simplest possible case $y = y' = 0$, $b_k = 1$, $\lambda_k = \lambda$ starting from (13) compared with that starting from (15).

Assuming that a simple explicit expression is desired for the weight polynomial then the utility of the CT theorem depends on and arises from three factors. The first problem is how to calculate the orthogonal polynomials. For various choices of the weights b_k and λ_k , the classical orthogonal polynomials are obtained and hence this problem has already been solved. However, for the applications mentioned earlier, which require ‘decorated’ weights, the polynomials³ do not fall into any of the classical classes. Thus, computing the polynomials becomes a problem in itself and is addressed by theorem 3.

The second problem is more subtle and is concerned with how the polynomials are represented. Whilst $P_k(x)$ is by construction a polynomial this is not necessarily how it is first represented. For example, consider a Chebyshev type polynomial, which satisfies the constant coefficient recurrence relation

$$S_{k+1}(x) = xS_k(x) - S_{k-1}(x), \quad S_1(x) = x, \quad S_0(x) = 1. \quad (16)$$

This recurrence is easily solved by substituting the usual trial solution $S_k = v^k$ with v a constant, leading immediately to a solution in the form

$$S_k(x) = \frac{(x + \sqrt{x^2 - 4})^{k+1} - (x - \sqrt{x^2 - 4})^{k+1}}{2^{k+1}\sqrt{x^2 - 4}}. \quad (17)$$

Whilst this is a polynomial in x , in this form it is not *explicitly* a polynomial⁴ as it is written in terms of the branches of an algebraic function. The representation of the polynomials is important as it strongly influences the third problem, that of computing the constant term (or residue). If the polynomials are explicitly polynomials (rather than, say, represented by algebraic functions), then the obvious way of computing the Laurent expansion, and hence residue, is via a geometric expansion of the denominator. Whilst in principal this can always be done for a rational function, the simpler the denominator polynomials the simpler the weight polynomial expression—in particular we would like as few summands as possible, preferably a number that does not depend on L , the height of the strip.

The fact that this can be achieved for the applications studied here shows the advantage of the CT theorem in this context over, say, the Rogers formula [26] (see proposition 3A of [19]), in which the weight polynomial is always expressed as an L -fold sum, no matter how small the set of decorated weights.

3. A paving theorem

Our second theorem is used to find explicit expressions for the orthogonal polynomials which are useful to our applications. These are polynomials arising from problems where the number of decorated weights is fixed (i.e. independent of L). Theorem 3, of which we make extensive use, expresses the orthogonal polynomial of the decorated weight problem in terms of the orthogonal polynomials of the problem with no decorated weights (i.e. Chebyshev type polynomials).

Theorem 3 (Paving).

(i) For each $c \in \{1, 2, \dots, k - 1\}$, we have an ‘edge-cutting’ identity

$$P_k^{(j)}(x) = P_c^{(j)}(x)P_{k-c}^{(j+c)}(x) - \lambda_{c+j}P_{c-1}^{(j)}(x)P_{k-c-1}^{(j+c+1)}(x) \quad (18a)$$

³ They may be thought of as ‘perturbed’ Chebyshev polynomials.

⁴ The square roots can of course be Taylor expanded to show explicitly that it is a polynomial.

and a ‘vertex-cutting’ identity,

$$P_k^{(j)}(x) = (x - b_{c+j})P_c^{(j)}(x)P_{k-c-1}^{(j+c+1)}(x) - \lambda_{j+c+1}P_c^{(j)}(x)P_{k-c-2}^{(j+c+2)}(x) - \lambda_{c+j}P_{c-1}^{(j)}(x)P_{k-c-1}^{(j+c+1)}(x), \tag{18b}$$

where $P_k^{(j)}(x)$ satisfies (12).

(ii) Fix j and k . Let $|\mathcal{D}_b|$ and $|\mathcal{D}_\lambda|$ be the number of decorated ‘across’ and ‘down’ steps, respectively, whose indices are strictly between $j - 1$ and $j + k$. Let $d = |\mathcal{D}_\lambda| + |\mathcal{D}_b|$. Then

$$P_k^{(j)}(x) = \sum_{j=1}^{j_{\max}} a_j \prod_{i=1}^{i_{\max}} S_{k_{j,i}}(x), \tag{19}$$

where

$$1 \leq j_{\max} \leq 2^{|\mathcal{D}_\lambda|} 3^{|\mathcal{D}_b|}, \quad 1 \leq i_{\max} \leq d + 1; \tag{20}$$

and $k_{j,i}$ is a positive integer valued function; decorations are all contained in the coefficient a_j ’s, and the $S_{k_{j,i}}(x)$ ’s are the background weight-dependent Chebyshev orthogonal polynomials satisfying

$$S_{k+1}(x) = (x - b) S_k(x) - \lambda S_{k-1}(x), \tag{21}$$

with $S_1(x) = x - b$, $S_0(x) = 1$ and $\lambda \neq 0$.

We do not give an explicit expression for the $k_{j,i}$ as it is strongly dependent on the sets \mathcal{D}_b and \mathcal{D}_λ . They are however simple to compute in any particular case, for example, see (58) in the application, section 7. The significance of (19) is that it shows that the decorated polynomials $P_k^{(j)}$ can be explicitly expressed in terms of the undecorated (i.e. Chebyshev) polynomials $S_k^{(m)}(x)$.

The first part of the paving theorem 3 follows immediately from an ‘edge-cutting’ and a ‘vertex-cutting’ technique, respectively, applied to Viennot’s paving representation of orthogonal polynomials, which we describe in section 7. This geometric way of visualizing an entire recurrence in one picture is powerful; from it we see part (i) of the theorem as a gestalt, so in practice we do not need to remember the algebraic expressions but may work with paving diagrams directly. Part (ii) follows immediately by induction on part (i).

4. Proof of Viennot’s theorem by transfer matrix diagonalization

In this section, we state a new proof of Viennot’s theorem. This proof starts with the transfer matrix for the Motzkin path (see section 4.7 of [27] for an explanation of the transfer matrix method) and proceeds by diagonalizing the matrix. As is well known this requires summing an expression over all the eigenvalues of the matrix. The eigenvalue sum is a sum over the zeros of a particular orthogonal polynomial. This sum can be done for the most general orthogonal polynomial even though the zeros are not explicitly known. We do this in two ways, the first uses two classical results (lemma 4.2 and lemma 4.3) and a paving polynomial identity. The essential idea is to replace the sum by a sum over residues and this residue sum can then be replaced by a single residue at infinity. The second proof uses partial fractions.

The transfer matrix, T_L , for paths in a strip of height L , is a square matrix of order $L + 1$ such that the (y', y) th entry of the t th power of the matrix gives the weight polynomial for paths of length t , i.e.

$$Z_t(y', y; L) = (T_L^t)_{y',y}. \tag{22}$$

For Motzkin paths with weights (4) the transfer matrix is the Jacobi matrix

$$T_L := \begin{pmatrix} b_0 & 1 & 0 & \dots & & & 0 \\ \lambda_1 & b_1 & 1 & 0 & \dots & & 0 \\ 0 & \lambda_2 & b_2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_{L-2} & b_{L-2} & 1 & 0 \\ 0 & & \dots & 0 & \lambda_{L-1} & b_{L-1} & 1 \\ 0 & & & \dots & 0 & \lambda_L & b_L \end{pmatrix}. \tag{23}$$

The standard path length generating function for such paths, with specified initial height y' and final height y , is given in terms of powers of the transfer matrix as

$$M_L(y', y; x) := \sum_{t \geq 0} Z_t(y', y; L)x^t = \sum_{t \geq 0} (T_L^t)_{y',y} x^t \tag{24}$$

which is convergent for $|x|$ smaller than the reciprocal of the absolute value of the largest eigenvalue of T_L .

The details of the proof are as follows. We evaluate $Z_t(y', y; L) = (T_L^t)_{y',y}$ by diagonalization

$$T_L^t = V D_L^t U \tag{25}$$

with $D_L = \text{diag}(x_0, x_1, \dots, x_L)$ a diagonal matrix of eigenvalues of T_L and V and U respectively matrices of right eigenvectors as columns and left eigenvectors as rows normalized such that

$$UV = I, \tag{26}$$

where I is the unit matrix. One may check that this diagonalization is achieved by setting the i th column of V equal to the transpose of

$$\mathbf{v}(x_{i-1}) = (P_0(x_{i-1}), P_1(x_{i-1}), \dots, P_L(x_{i-1})) \tag{27}$$

and the i th row of U equal to

$$\mathbf{u}(x_{i-1}) = \frac{\lambda_1 \dots \lambda_L}{P'_{L+1}(x_{i-1})P_L(x_{i-1})} \left(\frac{P_0(x_{i-1})}{1}, \frac{P_1(x_{i-1})}{\lambda_1}, \frac{P_2(x_{i-1})}{\lambda_1 \lambda_2}, \dots, \frac{P_L(x_{i-1})}{\lambda_1 \dots \lambda_L} \right), \tag{28}$$

where the set of eigenvalues $\{x_i\}_{i=0}^L$ is determined by

$$P_{L+1}(x_i) = 0, \tag{29}$$

with the orthogonal polynomial P_{L+1} given by the three-term recurrence (12). Orthogonality of left with right eigenvectors of the Jacobi matrix (23) follows by using the Christoffel–Darboux theorem for orthogonal polynomials. Equation (26) then follows as (29) gives $L + 1$ distinct zeros and hence $L + 1$ distinct eigenvalues and hence $L + 1$ linearly independent eigenvectors.

For simplicity in the following we only consider the case $y' \leq y$ in which case the $h_{y',y}$ factor is one—it is readily inserted for the case $y' > y$. Thus, multiplying out equation (25) and extracting the (y', y) th entry, we have

$$(T_L^t)_{y',y} = (\lambda_{y+1} \dots \lambda_L) \sum_{i=0}^L \frac{x_i^t P_{y'}(x_i) P_y(x_i)}{P'_{L+1}(x_i) P_L(x_i)}. \tag{30}$$

Note that $P'_{L+1}(x_i) \neq 0$ and $P_L(x_i) \neq 0$ by the interlacing theorem for orthogonal polynomials, so that all the terms in the sum are finite. Since Viennot’s theorem does not have a product

of polynomials in the denominator we need to simplify (30), which is achieved by using the following lemma.

Lemma 4.1. *Let x_i be a zero of $P_{L+1}(x)$. Then*

$$\lambda_{y+1} \dots \lambda_L P_y(x_i) = P_L(x_i) P_{L-y}^{(y+1)}(x_i). \tag{31}$$

This lemma follows directly from the edge-cutting identity (18a) by choosing $k = L + 1$, $j = c$ and $c = L - h$ together with the assumption that $P_{L+1}(x_i) = 0$, to obtain a family of identities parametrized by h , which are then iterated with $h = 0, 1, \dots, L - y - 1$.

Applying lemma 4.1 to (30) gives the following basic expression for the weight polynomial resulting from the transfer matrix:

$$(T_L^t)_{y',y} = \sum_{i=0}^L \frac{x_i^t P_{y'}(x_i) P_{L-y}^{(y+1)}(x_i)}{P_{L+1}'(x_i)}. \tag{32}$$

Note, this use of the transfer matrix leads *first* to an expression for the weight polynomial, to get to Viennot’s theorem we still need to generate on the path length and also simplify the sum over zeros. The former problem is trivial, the latter not. We do the sum over zeros in two ways, first by using a contour integral representation and secondly by using partial fractions.

4.1. Using a contour integral representation

To eliminate the derivative in the denominator of (32) we use the following Lemma.

Lemma 4.2. *Let $P(z)$ and $Q(z)$ be polynomials in a complex variable z and $P(z_i) = 0$, then*

$$m_i Q(z_i) = \text{Res} \left[Q(z) \frac{P'(z)}{P(z)}, \{z, z_i\} \right]. \tag{33}$$

where m_i is the multiplicity of the root z_i , and $P'(z)$ is the derivative with respect to z .

The following Lemma allows us to replace a residue sum with a residue at infinity.

Lemma 4.3. *Let $P(z)$ and $Q(z)$ be polynomials in a complex variable z , then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{P(z)}{Q(z)} dz = \sum_{z_i \in A} \text{Res} \left[\frac{P(z)}{Q(z)}, \{z, z_i\} \right] \tag{34a}$$

$$= \text{Res} \left[\frac{P(1/z)}{z^2 Q(1/z)}, \{z, 0\} \right], \tag{34b}$$

where γ is a simple closed anticlockwise-oriented contour enclosing all the zeros of $Q(z)$ and A is the set of zeros of $Q(z)$.

Note, (34) simply states that the sum of all residues of a rational function, including that at infinity, is zero. These lemmas are proved in most books on complex variables—see [5] for example.

Now use lemma 4.2 to get rid of the derivative in the denominator ($m_i = 1$ as all zeros of orthogonal polynomials are simple) to produce

$$(T_L^t)_{y',y} = \sum_{\{x_i | P_{L+1}(x_i)=0\}} \text{Res} \left[\frac{x_i^t P_{y'}(x) P_{L-y}^{(y+1)}(x)}{P_{L+1}(x)}, \{x, x_i\} \right]. \tag{35}$$

Applying lemma 4.3 to sum over the zeros, gives the weight polynomial as a single residue (or constant term)

$$(T_L^t)_{y',y} = \text{Res} \left[\frac{P_{y'}(1/x) P_{L-y}^{(y+1)}(1/x)}{x^{t+2} P_{L+1}(1/x)}, \{x, 0\} \right]. \tag{36}$$

As noted above, a factor $h_{y',y}$ needs to be inserted in the numerator for the case $y' > y$. Comparing this form with (15) and changing to the reciprocal polynomials (6) give Viennot’s theorem. Thus, we see the change to the reciprocal $1/x$ in this context corresponds to switching from a sum of residues to a single residue at infinity.

4.2. Using partial fractions

We can also derive a residue or constant term expression for the weight polynomial, (36), without invoking either of the calculus lemmas 4.2 or lemma 4.3 by using a partial fraction expansion of a rational function. In particular, if we have the rational function

$$G(x) = \frac{Q(x)}{T(x)},$$

with Q and T polynomials of degrees a and b , respectively, and $a < b$. Assuming that $T(x)$ is monic and has simple zeros $T(x_i) = 0$ we have $T(x) = \prod_i (x - x_i)$ and thus we have, using standard methods, the partial fraction expansion

$$G(x) = \sum_i \frac{Q(x_i)}{T'(x_i)} \frac{1}{x - x_i},$$

where $T'(x)$ is the derivative of $T(x)$ and thus

$$G(1/x) = \sum_i \frac{Q(x_i)}{T'(x_i)} \frac{x}{1 - x_i x}. \tag{37}$$

Geometric expansion of each term gives us the coefficient of x^n in $G(1/x)$ as

$$[x^n]G(1/x) = \sum_i x_i^{n-1} \frac{Q(x_i)}{T'(x_i)}. \tag{38}$$

If we now compare (32) with (38) we see that $Q \rightarrow P_{y'} P_{L-y}^{(y+1)}$, $T \rightarrow P_{L+1}$ and $t = n - 1$; thus we get

$$\begin{aligned} (T_L^t)_{y',y} &= [x^{t+1}] \frac{P_{y'}(1/x) P_{L-y}^{(y+1)}(1/x)}{P_{L+1}(1/x)} \\ &= \text{Res} \left[\frac{1}{x^{t+2}} \frac{P_{y'}(1/x) P_{L-y}^{(y+1)}(1/x)}{P_{L+1}(1/x)}, \{x, 0\} \right]. \end{aligned} \tag{39}$$

Note that the orthogonal polynomials satisfy the conditions required for the existence of (37). Thus, we see that the sum over the zeros (i.e. eigenvalues) in (32) is actually a term in the geometric expansion, or Taylor series, of a partial fraction expansion and hence ‘summed’ by reverting the expansion back to the rational function it arose from.

5. Proof of the constant term theorem

For the proof of the CT theorem we use the following: residue ‘change of variable’ lemma.

Lemma 5.1 (Residue change of variable). *Let $f(x)$ and $r(x)$ be the functions which have Laurent series about the origin, and the Laurent series of $r(x)$ has the property that $r(x) = x^k g(x)$ with $k > 0$ and $g(x)$ has a Taylor series $g(x) = \sum_{n \geq 0} a_n x^n$ such that $a_0 \neq 0$. Then*

$$\text{Res}[f(x), x] = \frac{1}{k} \text{Res} \left[f(r(z)) \frac{dr}{dz}, z \right], \tag{40}$$

where $\text{Res}[f(x), x]$ denotes the residue of $f(x)$ at $x = 0$.

The lemma appears to have been proved first by Jacobi [23]. A proof may also be found in Goulden and Jackson [21], theorem 1.2.2. Note, the condition on $r(x)$ is equivalent to the requirement that the Laurent series of the inverse function of $r(x)$ exist. To prove the CT theorem we start with Viennot’s theorem and use the simple fact that the coefficient of x^t in the Taylor series for $f(x)$ is given by

$$\text{CT}_x \left[\frac{1}{x^t} f(x) \right] = \text{Res} \left[\frac{1}{x^{t+1}} f(x) \right] \tag{41}$$

which gives (15), that is,

$$Z_t(y', y; L) = \text{CT}_x \left[\frac{1}{x^{t+1}} x^{y-y'} \frac{R_{y'}(x) h_{y',y} R_{L-y}^{(y+1)}(x)}{R_{L+1}(x)} \right]. \tag{42}$$

Now consider the change of variable defined by

$$x(\rho) = \frac{\rho}{\rho^2 + b\rho + \lambda}, \quad \lambda \neq 0, \tag{43}$$

which has the Taylor expansion about the origin

$$x(\rho) = \frac{\rho}{\lambda} - \frac{b\rho^2}{\lambda^2} + O(\rho^3)$$

and thus $x(\rho)$ satisfies the conditions of lemma 5.1 with $k = 1$. Note, (43) is the reciprocal of the change given in (11). Thus, from (42) and using lemma 5.1 with the change of variable (43) and derivative

$$\frac{d}{d\rho} x(\rho) = \frac{\lambda\rho^{-2} - 1}{(\rho + b + \lambda\rho^{-1})^2}, \tag{44}$$

we get the CT theorem result (13) in terms of the Laurent polynomials defined by (10).

We make two remarks. The first is the primary reason for the change of variable in this instance is that it ‘gets rid of the square roots’ such as those that appear in the representation (17) and hence changes a representation of a Chebyshev type polynomial in terms of algebraic functions to an explicit Laurent polynomial form.

For the second remark we note that for this proof of the CT theorem we could equally well have started just before the end of the diagonalization proof of the Viennot theorem, i.e. equation (36), and proceeded with the residue change of variable (43). In other words, the transfer matrix diagonalization naturally ends with an expression for the weight polynomial—this has to be generated on before getting to Viennot’s theorem, whilst the first step in this CT proof, i.e. equation (42), is to undo this generating step.

6. Proof of the paving theorem

The paving interpretation of the orthogonal polynomial three-term recurrence relation was introduced by Viennot [29]. We will use this interpretation as the primary means of proving theorem 3. First we define several terms associated with pavings. A *path graph* is any graph

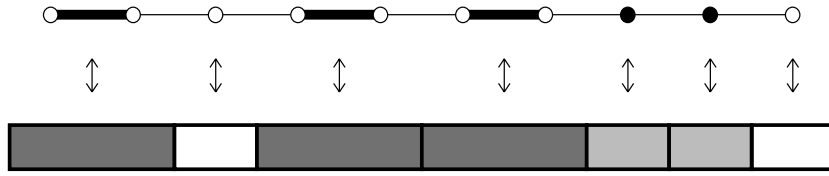


Figure 2. The upper part of this diagram shows a paving. The lower part indicates the reason for calling it a ‘paving’, as it is in bijection with a more standard ‘paving’ or ‘tiling’ diagram, where long and short tiles are used, the short tiles being of two possible colours.

isomorphic to a graph with vertex set $\{v_i\}_{i=0}^k$ and edge set $\{v_i v_{i+1}\}_{i=0}^{k-1}$. A *monomer* is a distinguished vertex in a graph. A *dimer* is a distinguished edge (pair of adjacent vertices). A *non-covered vertex* is a vertex which occurs in neither a monomer nor a dimer. A *paver* is any of the three possibilities: monomer, dimer or non-covered vertex. A *paving* is a collection of pavers on a path graph such that no two pavers share a vertex. We say that a paving is of *order* k if it occurs on a path graph with k vertices. An example of a paving of order 10 is shown in figure 2. *Weighted pavings* are pavings with weights associated with each paver. We will need pavers with shifted indices in order to calculate the shifted paving polynomials that occur in theorem 3. Thus, the *weight of a paver* α with *shift* j is defined as follows:

$$w_j(\alpha) = \begin{cases} x & \text{the paver } \alpha \text{ is a non-covered vertex} \\ -b_{i+j} & \text{the paver } \alpha \text{ is the monomer } v_i. \\ -\lambda_{i+j} & \text{the paver } \alpha \text{ is the dimer } e_i \end{cases} \quad (45)$$

The *weight of a paving* is defined to be the product of the weights of the pavers that comprise it, i.e.

$$w_j(p) = \prod_{\alpha \in p} w_j(\alpha), \quad (46)$$

for p a paving. It is useful to distinguish two kinds of paving. Pavings containing only non-covered vertices and dimers are called *Ballot pavings*; those also containing monomers are called *Motzkin pavings*. A *paving set* is the collection of all pavings (of either Ballot or Motzkin type) on a path graph of given size. We write

$$\mathcal{P}_k^{\text{Bal}} = \{p \mid p \text{ is a Ballot paving of order } k\} \quad (47)$$

$$\mathcal{P}_k^{\text{Motz}} = \{p \mid p \text{ is a Motzkin paving of order } k\}. \quad (48)$$

When it is clear by context whether we refer to sets of Ballot or Motzkin pavings, the explanatory superscript is omitted.

A *paving polynomial* is a sum over weighted pavings defined by

$$P_k^{(j)}(x) = \sum_{p \in \mathcal{P}_k} w_j(p), \quad (49)$$

with \mathcal{P}_k being either $\mathcal{P}_k^{\text{Bal}}$ or $\mathcal{P}_k^{\text{Motz}}$ and weights as in equation (45). If $k = 0$ then we define $P_0^{(j)}(x) := 1 \forall j$. The diagrammatic representation for a paving polynomial on a paving set of the Ballot type is

$$P_k^{(j)}(x) \leftarrow \begin{array}{c} \circlearrowleft \quad \overset{-\lambda_{j+1}}{\circlearrowright} \quad \circlearrowleft \quad \overset{-\lambda_{j+2}}{\circlearrowright} \quad \circlearrowleft \quad \dots \quad \circlearrowleft \quad \overset{-\lambda_{j+k-2}}{\circlearrowright} \quad \circlearrowleft \quad \overset{-\lambda_{j+k-1}}{\circlearrowright} \quad \circlearrowleft \\ v_0 \quad v_1 \quad v_2 \quad \dots \quad v_{k-3} \quad v_{k-2} \quad v_{k-1} \end{array} \quad (50)$$

where the question mark denotes that the edge can be either a dimer or not. The diagrammatic representation for a paving polynomial on a paving set of Motzkin type is

$$P_k^{(j)}(x) \leftarrow \begin{array}{c} -b_j \quad -\lambda_{j+1} \quad -b_{j+1} \quad -\lambda_{j+2} \quad -b_{j+2} \quad \dots \quad -b_{j+k-3} \quad -\lambda_{j+k-2} \quad -b_{j+k-2} \quad -\lambda_{j+k-1} \quad -b_{j+k-1} \\ \circ_{v_0} \text{---} \text{?} \circ_{v_1} \text{---} \text{?} \circ_{v_2} \text{---} \dots \text{---} \text{?} \circ_{v_{k-3}} \text{---} \text{?} \circ_{v_{k-2}} \text{---} \text{?} \circ_{v_{k-1}} \end{array} \quad (51)$$

Note, we overload the notation for $P_k^{(j)}(x)$ since we will use it for the set of pavings and the corresponding paving polynomial obtained by summing over all weighted pavings in the set.

Viennot [29] has shown that equation (49) satisfies (12) and hence $P_k^{(j)}(x)$ is an orthogonal polynomial. Ballot pavings correspond to the case $b_i = 0 \forall i$. An example of Ballot pavings and the resulting polynomial is

$$\begin{aligned} P_4^{(0)}(x) \leftarrow \begin{array}{c} -\lambda_1 \quad -\lambda_2 \quad -\lambda_3 \\ \circ_{v_0} \text{---} \text{?} \circ_{v_1} \text{---} \text{?} \circ_{v_2} \text{---} \text{?} \circ_{v_3} \end{array} &= \begin{array}{c} x \quad x \quad x \quad x \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \end{array} \\ &+ \begin{array}{c} -\lambda_1 \quad x \quad x \\ \circ \text{---} \text{---} \circ \text{---} \circ \end{array} \\ &+ \begin{array}{c} x \quad -\lambda_2 \quad x \\ \circ \text{---} \text{---} \circ \text{---} \circ \end{array} \\ &+ \begin{array}{c} x \quad -\lambda_3 \\ \circ \text{---} \circ \text{---} \text{---} \circ \end{array} \\ &+ \begin{array}{c} -\lambda_1 \quad -\lambda_3 \\ \circ \text{---} \text{---} \circ \text{---} \text{---} \circ \end{array} \\ &\rightarrow x^4 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + \lambda_1\lambda_3 \end{aligned}$$

and an example of a Motzkin paving set with the associated polynomial is

$$\begin{aligned} P_3^{(6)}(x) \leftarrow \begin{array}{c} b_6 \quad -\lambda_7 \quad b_7 \quad -\lambda_8 \quad b_8 \\ \circ_{v_0} \text{---} \text{?} \circ_{v_1} \text{---} \text{?} \circ_{v_2} \end{array} &= \begin{array}{c} x \quad x \quad x \\ \circ \text{---} \circ \text{---} \circ \end{array} + \begin{array}{c} -b_6 \quad -b_7 \quad x \\ \bullet \text{---} \bullet \text{---} \circ \end{array} + \begin{array}{c} -\lambda_7 \quad x \\ \circ \text{---} \text{---} \circ \end{array} \\ &+ \begin{array}{c} -b_6 \quad x \quad x \\ \bullet \text{---} \circ \text{---} \circ \end{array} + \begin{array}{c} -b_6 \quad -b_8 \\ \bullet \text{---} \bullet \text{---} \bullet \end{array} + \begin{array}{c} x \quad -\lambda_8 \\ \circ \text{---} \text{---} \circ \end{array} \\ &+ \begin{array}{c} x \quad -b_7 \quad x \\ \circ \text{---} \bullet \text{---} \circ \end{array} + \begin{array}{c} x \quad -b_7 \quad -b_8 \\ \circ \text{---} \bullet \text{---} \bullet \end{array} + \begin{array}{c} -b_6 \quad -\lambda_8 \\ \bullet \text{---} \text{---} \circ \end{array} \\ &+ \begin{array}{c} x \quad x \quad -b_8 \\ \circ \text{---} \circ \text{---} \bullet \end{array} + \begin{array}{c} -b_6 \quad -b_7 \quad -b_8 \\ \bullet \text{---} \bullet \text{---} \bullet \end{array} + \begin{array}{c} -\lambda_7 \quad -b_8 \\ \circ \text{---} \text{---} \bullet \end{array} \\ &\rightarrow x^3 - (b_6 + b_7 + b_8)x^2 \\ &\quad -(\lambda_7 + \lambda_8 - b_6b_7 - b_6b_8 - b_7b_8)x \\ &\quad -(b_6b_7b_8 - b_6\lambda_8 - b_8\lambda_7) \end{aligned}$$

The two identities of theorem 2 correspond to

- cutting at an arbitrary edge, and
- cutting at an arbitrary vertex.

In the first procedure we consider the edge e_c . This edge is either paved or not paved with a dimer. Equation (52) illustrates the division into these two cases, and the corresponding

polynomial identity. Note that the right-hand side of the expression obtained is a sum of the products of smaller order polynomials such that the weight ‘ $-\lambda_{c+j}$ ’ associated with an edge e_c occurs explicitly as a coefficient, and is not hidden inside any of the smaller order polynomials.

$$\begin{aligned}
 P_k^{(j)}(x) &\leftarrow \begin{array}{c} \textcircled{v_0} \text{---} \textcircled{v_{c-2}} \text{---} \textcircled{v_{c-1}} \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} \text{---} \textcircled{v_{k-1}} \\ \begin{array}{ccccccc} -b_j & & -b_{c+j-2} & -\lambda_{c+j-1} & -b_{c+j-1} & -\lambda_{c+j} & -b_{c+j} & -\lambda_{c+j+1} & -b_{c+j+1} & & -b_{j+k-1} \end{array} \end{array} \\
 &= \begin{array}{c} \textcircled{v_0} \text{---} \textcircled{v_{c-2}} \text{---} \textcircled{v_{c-1}} \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} \text{---} \textcircled{v_{k-1}} \\ \begin{array}{ccccccc} -b_j & & -b_{c+j-2} & -\lambda_{c+j-1} & -b_{c+j-1} & & -b_{c+j} & -\lambda_{c+j+1} & -b_{c+j+1} & & -b_{j+k-1} \end{array} \end{array} \\
 &+ \begin{array}{c} \textcircled{v_0} \text{---} \textcircled{v_{c-2}} \text{---} \textcircled{v_{c-1}} \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} \text{---} \textcircled{v_{k-1}} \\ \begin{array}{ccccccc} -b_j & & -b_{c+j-2} & & -\lambda_{c+j} & & -b_{c+j+1} & & -b_{c+j+1} & & -b_{j+k-1} \end{array} \end{array} \\
 &\rightarrow P_c^{(j)}(x)P_{k-c}^{(c+j)}(x) - \lambda_{c+j}P_{c-1}^{(j)}(x)P_{k-c-1}^{(c+j+1)}(x) \tag{52}
 \end{aligned}$$

The second procedure is to cut at an arbitrary vertex ‘ v_c ’. In this procedure the cases to consider are v_c is non-covered, v_c is a monomer, v_c is the leftmost vertex of a dimer and v_c is the rightmost vertex of a monomer. These four cases are shown in equation (53), and the resulting identity gives $P_k^{(j)}(x)$ as a sum of four terms, each of which contains a product of two smaller order polynomials. The vertex weight ‘ $-b_{c+j}$ ’ occurs explicitly as a coefficient and is not hidden in any of the smaller order polynomials.

$$\begin{aligned}
 P_k^{(j)}(x) &\leftarrow \begin{array}{c} \textcircled{v_0} \text{---} \textcircled{v_{c-2}} \text{---} \textcircled{v_{c-1}} \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} \text{---} \textcircled{v_{k-1}} \\ \begin{array}{ccccccc} -b_j & & -b_{c+j-2} & -\lambda_{c+j-1} & -b_{c+j-1} & -\lambda_{c+j} & -b_{c+j} & -\lambda_{c+j+1} & -b_{c+j+1} & & -b_{j+k-1} \end{array} \end{array} \\
 &= \begin{array}{c} \textcircled{v_0} \text{---} \textcircled{v_{c-2}} \text{---} \textcircled{v_{c-1}} \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} \text{---} \textcircled{v_{k-1}} \\ \begin{array}{ccccccc} -b_j & & -b_{c+j-2} & -\lambda_{c+j-1} & -b_{c+j-1} & x & -b_{c+j+1} & & -b_{c+j+1} & & -b_{j+k-1} \end{array} \end{array} \\
 &+ \begin{array}{c} \textcircled{v_0} \text{---} \textcircled{v_{c-2}} \text{---} \textcircled{v_{c-1}} \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} \text{---} \textcircled{v_{k-1}} \\ \begin{array}{ccccccc} -b_j & & -b_{c+j-2} & -\lambda_{c+j-1} & -b_{c+j-1} & \bullet & -b_{c+j+1} & & -b_{c+j+1} & & -b_{j+k-1} \end{array} \end{array} \\
 &+ \begin{array}{c} \textcircled{v_0} \text{---} \textcircled{v_{c-2}} \text{---} \textcircled{v_{c-1}} \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} \text{---} \textcircled{v_{k-1}} \\ \begin{array}{ccccccc} -b_j & & -b_{c+j-2} & -\lambda_{c+j-1} & -b_{c+j-1} & \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} & -\lambda_{c+j+1} & & -b_{c+j+2} & -b_{j+k-1} \end{array} \end{array} \\
 &+ \begin{array}{c} \textcircled{v_0} \text{---} \textcircled{v_{c-2}} \text{---} \textcircled{v_{c-1}} \text{---} \textcircled{v_c} \text{---} \textcircled{v_{c+1}} \text{---} \textcircled{v_{k-1}} \\ \begin{array}{ccccccc} -b_j & & -b_{c+j-2} & & -\lambda_{c+j} & & -b_{c+j+1} & & -b_{c+j+1} & & -b_{j+k-1} \end{array} \end{array} \\
 &\rightarrow (x - b_{c+j})P_c^{(j)}(x)P_{k-c-1}^{(c+j+1)}(x) - \lambda_{c+j+1}P_c^{(j)}(x)P_{k-c-2}^{(c+j+2)}(x) \\
 &\quad - \lambda_{c+j}P_{c-1}^{(j)}(x)P_{k-c-1}^{(c+j+1)}(x) \tag{53}
 \end{aligned}$$

These two ‘cutting’ procedures, as shown in diagrams (52) and (53), prove part (i) of theorem 3. Finally, part (ii) of the theorem follows immediately from part (i) by induction on the number of decorations of each of the two possible kinds (‘across step’ and ‘down step’).

We note that the proof of theorem 3 shows that the upper bounds given in equations (20) are tight only when the decorations are well separated from each other as well as from the ends of the diagram. Otherwise, pulling out a given decoration as a coefficient may pull out a neighbouring decoration in the same procedure, meaning that fewer terms are needed. Thus, we can deal more efficiently with decorated weightings in which collections of decorations are bunched together, as illustrated in the examples in section 7.

7. Applications

We now consider two applications. The first is the DiMazio and Rubin problem discussed in the introduction. Solving this model corresponds to determining the Ballot path weight polynomials with just one upper and one lower decorated weight. The second application is an extension of that problem in which two upper and two lower edges now carry decorated weights.

7.1. Two decorated weights

For the DiMazio and Rubin problem we need to compute the Ballot path weight polynomials with weights given by (54). A partial solution was given in [12] for the case when the upper weight is a particular function of the lower one i.e. $\kappa + \omega = \kappa\omega$. The first general solution was published in [7] in 2006. The solution we now give is an improvement on [7] (which was based on a precursor to this method) as it contains fewer summations.

Theorem 4. *Let $Z_{2r}(\kappa, \omega; L)$ be the weight polynomial for Dyck paths of length $2r$ confined to a strip of height L , and with weights (see (4)), $b = \lambda = 1$, $\hat{b}_i = 0 \forall i$ and*

$$\hat{\lambda}_i = \begin{cases} \omega - 1 & \text{if } i = L \\ \kappa - 1 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{54}$$

Then

$$Z_{2r}(\kappa, \omega; L) = \text{CT} \left[(\rho + \rho^{-1})^{2r} (1 - \rho^2) \frac{A\rho^L - B\rho^{-L}}{AC\rho^L - BD\rho^{-L}} \right], \tag{55}$$

where

$$A = \rho^2 - \hat{\omega} \tag{56a}$$

$$B = 1 - \hat{\omega}\rho^2 \tag{56b}$$

$$C = \rho^2 - \hat{\kappa} \tag{56c}$$

$$D = 1 - \hat{\kappa}\rho^2, \tag{56d}$$

$\hat{\kappa} := \kappa - 1$ and $\hat{\omega} := \omega - 1$.

An example of the paths referred to in theorem 4 is given in figure 3. Theorem 4 is derived using theorem 2 and theorem 3. We shall show the derivation of the denominator of (55) but omit the details for the numerator which may be similarly derived.

We work with polynomials in the x variable, and then use equation (10) to get the expression in terms of ρ . First we write $P_{L+1}(x)$ for our given weighting as in equation (57)

$$P_{L+1}(x) = \underset{0}{\circ} \overset{-\kappa}{\text{---}} \underset{1}{\circ} \overset{-1}{\text{---}} \underset{2}{\circ} \overset{-1}{\text{---}} \underset{3}{\circ} \overset{-1}{\text{---}} \underset{4}{\circ} \text{---} \dots \text{---} \underset{L-4}{\circ} \overset{-1}{\text{---}} \underset{L-3}{\circ} \overset{-1}{\text{---}} \underset{L-2}{\circ} \overset{-1}{\text{---}} \underset{L-1}{\circ} \overset{-\omega}{\text{---}} \underset{L}{\circ} \tag{57}$$

Next, we indicate how to expand equation (55) to give an expansion for the weight polynomial in terms of binomials. Our particular expression is obtained via straightforward application of geometric series and binomial expansions, with the resulting formula containing a fivefold sum. It is certainly possible to do worse than this and obtain more sums by making less judicious choices of representations while carrying out the expansion, but it seems unlikely that elementary methods can yield a smaller than fivefold sum for this problem.

First, manipulate the fractional part of equation (55) into a form in which geometric expansion of the denominator is natural, as in line (63) below. Do the geometric expansion and multiply out to give two terms: extract the $m = 0$ case from the first term and shift the index of summation in the second to give line (65).

$$\frac{A\rho^L - B\rho^{-L}}{AC\rho^L - BD\rho^{-L}} = \frac{\frac{1}{D} - \frac{A}{BD}\rho^{2L}}{1 - \frac{AC}{BD}\rho^{2L}} \tag{63}$$

$$= \left(\frac{1}{D} - \frac{A}{BD}\rho^{2L}\right) \sum_{m=0}^{\infty} \left(\frac{AC}{BD}\right)^m \rho^{2mL} \tag{64}$$

$$= \frac{1}{D} + \sum_{m=1}^{\infty} \frac{A^m C^m}{B^m D^{m+1}} \rho^{2mL} - \sum_{m=1}^{\infty} \frac{A^m C^{m-1}}{B^m D^m} \rho^{2mL}. \tag{65}$$

We next find the CT, separately, of each of the three terms multiplied by series for $(\rho + \rho^{-1})^{2r}(1 - \rho^2)$. The first gives

$$\text{CT} \left[(\rho + \rho^{-1})^{2r} (1 - \rho^2) \frac{1}{D} \right] \tag{66}$$

$$= \text{CT} \left[\left(\sum_{u=0}^{2r} \binom{2r}{u} \rho^{2r-2u} \right) (1 - \rho^2) \left(\sum_{m=0}^{\infty} \hat{\kappa}^m \rho^{2m} \right) \right] \tag{67}$$

$$= \sum_{m=0}^{\infty} \hat{\kappa}^m \text{CT} \left[\sum_{u=0}^{2r} \binom{2r}{u} (\rho^{2r-2u+2m} - \rho^{2r-2u+2m+2}) \right] \tag{68}$$

$$= \sum_{m=0}^{\infty} \hat{\kappa}^m \left[\binom{2r}{r+m} - \binom{2r}{r+m+1} \right] \tag{69}$$

$$= \sum_{m=0}^{\infty} C_{r;r-m} \hat{\kappa}^m, \tag{70}$$

where $C_{n;k} := \binom{2n}{k} - \binom{2n}{k-1}$. The second term is expanded similarly, with the positive powers of A and C and the negative powers of B and D each contributing a single sum, which, when concatenated with the original sum over m , creates a fivefold sum altogether. The third term generates a similar fivefold sum, and this difference of a pair of fivefold sums is combined into one in the final expression in theorem 5 below.

Theorem 5. *Let $Z_{2r}(\kappa, \omega; L)$ be as in theorem 4. Then*

$$\begin{aligned} Z_{2r}(\kappa, \omega; L) = & \sum_{m \geq 0} C_{r;r-m} \hat{\kappa}^m + \sum_{m \geq 1} \sum_{p_1, p_2 \geq 0} \sum_{s_1, s_2 = 0}^m (-1)^{s_1+s_2} \hat{\kappa}^{s_2+p_2} \hat{\omega}^{s_1+p_1} \\ & \times \binom{m}{s_1} \binom{m}{s_2} \binom{m-1+p_1}{p_1} \binom{m+p_2}{p_2} \left[C_{r;r-k-1} - \frac{m-s_2}{m+p_2} C_{r;r-k} \right], \end{aligned} \tag{71}$$

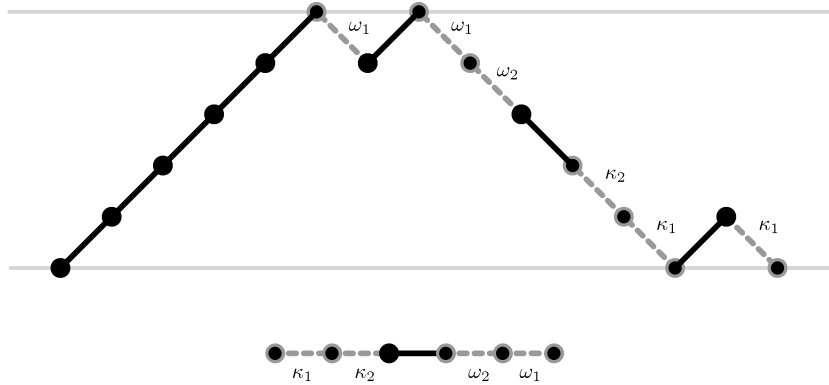


Figure 4. An example of a four-weight Dyck path (above) in a strip of height 5 and the corresponding paving problem (below).

where $k = p_1 + p_2 - s_1 - s_2 + (L + 2)m - 1$, and $C_{n;k}$ is the extended Catalan number,

$$C_{n;k} = \binom{2n}{k} - \binom{2n}{k-1}.$$

The binomial coefficient $\binom{n}{m}$ is assumed to vanish if $n < 0$ or $m < 0$ or $n < m$.

Note, when trying to rearrange (71) care should be taken when using any binomial identities because of the vanishing condition on the binomial coefficients—the support of any new expression must be the same as the support before (alternatively the upper limits of all of the summations must be precisely stated).

7.2. Four Decorated weights

This second problem is a natural generalization of the previous problem. We now have a pair of decorated weights in the pair of rows adjacent to each wall, as in figure 4. In the earlier DiMazio–Rubin problem, paths have been interpreted as polymers zig-zagging between comparatively large colloidal particles (large enough to be approximated by flat walls above and below) with an interaction occurring only upon contact between the surface and the polymer; this weighting scheme could be used to model such polymer systems, but now with a longer range interaction strength that varies sharply with separation from the colloid.

Theorem 6. Let $Z_{2r}(\kappa_1, \kappa_2, \omega_1, \omega_2; L)$ be the weight polynomial for Dyck paths of length $2r$ confined to a strip of height L , and with weights (see (4)), $b = \lambda = 1$, $\hat{b}_i = 0 \forall i$ and

$$\hat{\lambda}_i = \begin{cases} \omega_1 - 1 & \text{if } i = L \\ \omega_2 - 1 & \text{if } i = L - 1 \\ \kappa_2 - 1 & \text{if } i = 2 \\ \kappa_1 - 1 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (72)$$

Then

$$Z_{2r}(\kappa_1, \kappa_2, \omega_1, \omega_2; L) = \text{CT} \left[(\rho + \rho^{-1})^{2r} \left(\frac{AB\rho^L - \overline{AB}\rho^{-L}}{CB\rho^L - \overline{CB}\rho^{-L}} \right) (\rho^{-1} - \rho) \right] \quad (73)$$

where

$$A = 1 - \hat{\kappa}_2 \rho^{-2} \tag{74a}$$

$$\bar{A} = 1 - \hat{\kappa}_2 \rho^2 \tag{74b}$$

$$B = \rho - (\hat{\omega}_1 + \hat{\omega}_2) \rho^{-1} - \hat{\omega}_2 \rho^{-3} \tag{74c}$$

$$\bar{B} = \rho^{-1} - (\hat{\omega}_1 + \hat{\omega}_2) \rho - \hat{\omega}_2 \rho^3 \tag{74d}$$

$$C = \rho - (\hat{\kappa}_1 + \hat{\kappa}_2) \rho^{-1} - \hat{\kappa}_2 \rho^{-3} \tag{74e}$$

$$\bar{C} = \rho^{-1} - (\hat{\kappa}_1 + \hat{\kappa}_2) \rho - \hat{\kappa}_2 \rho^3 \tag{74f}$$

for $\hat{\kappa}_i := \kappa_i - 1$ and $\hat{\omega}_i := \omega_i - 1$.

An example of the paths referred to in the above theorem is given in figure 4.

The constant term expression in theorem 6 may be expanded in a similar fashion to that of theorem 4 to yield a ninefold sum. The fractional component of equation (73) may be written

$$\frac{AB\rho^L - \bar{A}\bar{B}\rho^{-L}}{CB\rho^L - \bar{C}\bar{B}\rho^{-L}} = \frac{\bar{A}}{\bar{C}} + \bar{A} \sum_{m=1}^{\infty} \frac{C^m B^m}{\bar{C}^{m+1} \bar{B}^m} \rho^{2mL} - A \sum_{m=1}^{\infty} \frac{C^{m-1} B^m}{\bar{C}^m \bar{B}^m} \rho^{2mL} \tag{75}$$

by a method precisely analogous to that applied in the two-weights' case. When multiplied by $(\rho + \rho^{-1})^{2r}(\rho^{-1} - \rho)$ and the constant term extracted, the initial term gives the double summation in the first term of equation (76) in theorem 7 below. The other two terms of equation (75) each give ninefold sums, as a consequence of the double sums yielded by each of the powers, positive and negative, of C, B, \bar{C} and \bar{B} . These two ninefold sums are combined in the second term of equation (76) in theorem 7.

Theorem 7. Let $Z_{2r}(\kappa_1, \kappa_2, \omega_1, \omega_2; L)$ be as in theorem 6. Then

$$\begin{aligned} Z_{2r}(\kappa_1, \kappa_2, \omega_1, \omega_2; L) &= \sum_{i \geq 0} \sum_{j=0}^i \binom{i}{j} (\hat{\kappa}_1 + \hat{\kappa}_2)^j \hat{\kappa}_2^{i-j} \\ &\times \left(\hat{\kappa}_2 \binom{2r}{u_0+2} - (\hat{\kappa}_2 + 1) \binom{2r}{u_0+1} + \binom{2r}{u_0} \right) \\ &+ \sum_{m \geq 1} \sum_{s_1=0}^m \sum_{i_1=0}^{s_1} \sum_{s_2=0}^m \sum_{i_2=0}^{s_2} \sum_{v_1 \geq 0}^{v_1} \sum_{j_1=0}^{v_1} \sum_{v_2 \geq 0}^{v_2} \sum_{j_2=0}^{v_2} \binom{s_1}{i_1} \binom{m}{s_2} \binom{s_2}{i_2} \binom{v_1}{j_1} \\ &\times \binom{v_2+m-1}{m-1} \binom{v_2}{j_2} (-1)^{s_1+s_2+i_1+i_2} \\ &\times \hat{\kappa}_2^{i_1+j_1} (\hat{\kappa}_1 + \hat{\kappa}_2)^{m+v_1-1-s_1-j_1} \hat{\omega}_2^{i_2+j_2} (\hat{\omega}_1 + \hat{\omega}_2)^{m+v_2-s_2-j_2} \\ &\times \left\{ \binom{m}{s_1} \binom{v_1+m}{m} (\hat{\kappa}_1 + \hat{\kappa}_2) \left(\hat{\kappa}_2 \binom{2r}{u_1+2} - (\hat{\kappa}_2 + 1) \binom{2r}{u_1+1} + \binom{2r}{u_1} \right) \right. \\ &\left. - \binom{m-1}{s_1} \binom{v_1+m-1}{m-1} \left(\hat{\kappa}_2 \binom{2r}{u_1-1} - (\hat{\kappa}_2 + 1) \binom{2r}{u_1} + \binom{2r}{u_1+1} \right) \right\} \tag{76} \end{aligned}$$

for $u_0 = r + 2i - j$ and $u_1 = r + mL + v_1 + v_2 + s_1 + s_2 + j_1 + j_2 - 2i_1 - 2i_2$.

Theorems 5 and 7 are to be compared with the Rogers formula below, which gives the weight polynomial as an order L -fold sum.

Theorem 8 (Rogers [26]). *Let Z_{2n} be the weight polynomial for the set of Dyck paths of length $2n$ with general down-step weighting (see (4)): $b = \lambda = 1$, $\hat{b}_i = 0 \forall i$ and $\hat{\lambda}_i = \kappa_i - 1$ in either a strip of height L or in the half plane (take $L = \infty$). Then the weight polynomial is given by*

$$Z_{2n}(\kappa_1, \kappa_2, \dots; L) = \sum_{l=0}^{\min\{n-1, L-1\}} s_l, \tag{77}$$

where s_l is the weight polynomial for that subset of paths in Z_{2n} which reach but do not exceed height $L + 1$. The s_l 's are given by

$$s_0 = \kappa_1^n, \tag{78}$$

and

$$s_l = \sum_{j_1=l}^{j_0-1} \sum_{j_2=l-1}^{j_1-1} \dots \sum_{j_{l-1}=1}^{j_{l-2}-1} \prod_{k=0}^{l-1} \binom{j_k - j_{k+2} - 1}{j_k - j_{k+1}} \kappa_1^{j_0-j_1} \kappa_2^{j_1-j_2} \dots \kappa_{l+1}^{j_l-j_{l+1}}, \tag{79}$$

for $l \geq 1$, with

$$j_0 := n, \tag{80}$$

$$j_{l+1} := 0. \tag{81}$$

We see that there is a trade-off between having comparatively few sums (compared with the width of the strip) but a complicated summand, as in theorems 5 and 7, versus having a simpler summand but the order of L sums irrespective of the number of decorations.

It would be an interesting piece of further research to see whether the solutions to the problems presented in section 7, containing as they do multiple alternating sums, are in some appropriate sense best possible or not. Another potentially useful area of further research would be an investigation of good techniques for extracting asymptotic information directly from the CT_ρ expression. It would also be interesting to have a pure algebraic formulation of this constant term method—one that does not rely on any residue theorems.

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